GENERALIZED JOSEPH'S DECOMPOSITIONS

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Abstract. We generalize the decomposition of $U_q(\mathfrak{g})$ introduced by A. Joseph in [5] and relate it, for $\mathfrak{g}$ semisimple, to the celebrated computation of central elements due to V. Drinfeld ([2]). In that case we construct a natural basis in the center of $U_q(\mathfrak{g})$ whose elements behave as Schur polynomials and thus explicitly identify the center with the ring of symmetric functions.

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1. Introduction and main results

1.1. Let H be an associative algebra with unity over a field $\Bbbk$ and let $\mathscr{C}$ be a full abelian subcategory closed under submodules of the category $H-\mathrm{Mod}$ of left H-modules. Suppose that we have a "finite duality" functor ${}^\star : \mathscr{C} \to \mathrm{Mod}-H$ with $V^\star \subseteq V^* = \mathrm{Hom}_\Bbbk(V, \Bbbk)$ (with equality if and only if V is finite dimensional) with its natural right H-module structure, such that the restriction of the evaluation pairing $\langle\cdot,\cdot\rangle_V : V \otimes V^* \to \Bbbk$ to $V \otimes V^\star$ is non-degenerate for all objects V in $\mathscr{C}$ (see §2.1 for the details). Following [4], we define $\beta_V : V \otimes_{D(V)} V^\star \to H^*$ where $D(V) = \mathrm{End}_H V^\star = (\mathrm{End}_H V)^{op}$ by

$$\beta_V(v \otimes f)(h) = \langle h \triangleright v, f\rangle_V = \langle v, f \triangleleft h\rangle_V, \qquad v \in V,\ f \in V^\star,\ h \in H,$$

where $\triangleright$ (respectively, $\triangleleft$) denotes the left (respectively, right) H-action. It is easy to see that β_V is well-defined. Set $H^*_V = \mathrm{Im}\,\beta_V$. Recall that $V \otimes V^\star$ and H^* are naturally H-bimodules. The following is essentially proved in [4, §3.1] and [3, Corollary 1.16]

Proposition 1.1. (a) *For all $V \in \mathscr{C}$, β_V is a homomorphism of H-bimodules and H^*_V depends only on the isomorphism class of V. Moreover, if $V, V' \in \mathscr{C}$ are simple and $H^*_V = H^*_{V'}$ then $V \cong V'$;*

(b) *$H^*_{V\oplus V'} = H^*_V + H^*_{V'}$ for all $V, V' \in \mathscr{C}$. In particular, $H^*_{V^{\oplus n}} = H^*_V$ for all $n \in \mathbb{N}$.*

(c) *If $V \otimes_{D(V)} V^\star$ is simple as an H-bimodule then β_V is injective.*

(d) *If V is simple finite dimensional then $V \otimes_{D(V)} V^\star$ is simple as an H-bimodule and hence β_V is injective.*

It is natural to call H^*_V a *generalized Peter-Weyl component*. Denote $H^*_{\mathscr{C}} = \sum_{[V]\in \mathrm{Iso}\,\mathscr{C}} H^*_V$ and $\underline{H}^*_{\mathscr{C}} = \bigoplus_{[V]\in \mathrm{Iso}^\circ\mathscr{C}} H^*_V$, where $\mathrm{Iso}\,\mathscr{C}$ (respectively, $\mathrm{Iso}^\circ\mathscr{C}$) is the set of isomorphism classes of objects (respectively, simple objects) in $\mathscr{C}$. By definition there is a natural homomorphism of H-bimodules $\underline{H}^*_{\mathscr{C}} \to H^*_{\mathscr{C}}$. Clearly, under the assumptions of Proposition 1.1(c) it is injective. Note that $H^*_{\mathscr{C}} = \sum_{[V]\in A} H^*_V$ for any subset A of $\mathrm{Iso}\,\mathscr{C}$ which generates it as an additive monoid. The following refinement of [4, Theorem 3.10] establishes the generalized Peter-Weyl decomposition.

Theorem 1.2. *Suppose that all objects in $\mathscr{C}$ have finite length. Then*

(a) *if $H^*_{\mathscr{C}} = \underline{H}^*_{\mathscr{C}}$ then $\mathscr{C}$ is semisimple;*

(b) *if $\mathscr{C}$ is semisimple and $V \otimes_{D(V)} V^\star$ is simple for every $V \in \mathscr{C}$ simple then $H^*_{\mathscr{C}} = \underline{H}^*_{\mathscr{C}}$.*

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1.2. Henceforth we denote by $\mathscr{C}^{fin}$ the full subcategory of $\mathscr{C}$ consisting of all finite dimensional objects. Clearly $V\otimes V^\star$, $V\in\mathscr{C}^{fin}$, is a unital algebra with the unity 1_V; set $z_V:=\beta_V(1_V)\in H^*_V$. For example, if $H=\Bbbk G$ for a finite group G then for any finite dimensional H-module V we have $z_V(g)=tr_V(g)$, $g\in G$ where tr_V denotes the trace of a linear endomorphism of V.

Given an H-bimodule B, define the subspace B^H of H-invariants in B by $B^H=\{b\in B\,:\,h\triangleright b=b\triangleleft h,\,\forall\,h\in H\}$ (B^H is sometimes referred to as the center of B). Clearly, $z_V\in(H^*_V)^H$, $z_V(1_H)=\dim_\Bbbk V\neq 0$ and $(H^*_V)^H=\Bbbk z_V$ if $\operatorname{End}_H V=\Bbbk\,\mathrm{id}_V$. Set $\mathcal{Z}_\mathscr{C}=\sum_{[V]\in\operatorname{Iso}\mathscr{C}}\mathbb{Z}z_V$. Given $V\in\mathscr{C}$, denote $|V|$ its image in the Grothendieck group $K_0(\mathscr{C})$ of $\mathscr{C}$. The following result contrasts sharply with Proposition 1.1 and Theorem 1.2 for non-semisimple $\mathscr{C}$.

Theorem 1.3. *Suppose that $\mathscr{C}=\mathscr{C}^{fin}$. Then the map $K_0(\mathscr{C})\to\mathcal{Z}_\mathscr{C}$ given by $|V|\mapsto z_V$, $[V]\in\operatorname{Iso}\mathscr{C}$ is an isomorphism of abelian groups.*

1.3. To introduce a multiplication on $\mathcal{Z}_\mathscr{C}\subset(H^*_\mathscr{C})^H\subset H^*_\mathscr{C}$, we assume henceforth that $H=(H,m,\Delta,\varepsilon)$ is a bialgebra and that $\mathscr{C}$ is a tensor subcategory of $H-\mathrm{Mod}$. Note that H^* is an algebra in a natural way. It is easy to see (Lemma 2.4) that $(H^*)^H$ is a subalgebra of H^*. We also assume that there is a natural isomorphism $(V\otimes V')^\star\cong V'^\star\otimes V^\star$ in $\mathrm{mod}-H$ for all $V,V'\in\mathscr{C}$.

Theorem 1.4. (a) *$H^*_V\cdot H^*_{V'}=H^*_{V\otimes V'}$ for all $V,V'\in\mathscr{C}$. In particular, $H^*_\mathscr{C}$ is a subalgebra of H^*;*

(b) *$z_V\cdot z_{V'}=z_{V\otimes V'}$ for all $V,V'\in\mathscr{C}^{fin}$. In particular, if $\mathscr{C}=\mathscr{C}^{fin}$ then $\mathcal{Z}_\mathscr{C}$ is a subring of $(H^*_\mathscr{C})^H$ and the map $K_0(\mathscr{C})\to\mathcal{Z}_\mathscr{C}$ from Theorem 1.3 is an isomorphism of rings.*

Thus, it is natural to regard $\mathcal{Z}_\mathscr{C}$ as the character ring of $\mathscr{C}$.

1.4. It turns out that we can transfer the above structures from $H^*_\mathscr{C}$ to H if $H=(H,m,\Delta,\varepsilon,S)$ is a Hopf algebra. For an H-bimodule B define left actions ad and $\diamond$ on B via $(\operatorname{ad}h)(b)=h_{(1)}\triangleright b\triangleleft S(h_{(2)})$ and $h\diamond b=S^2(h_{(2)})\triangleright b\triangleleft S(h_{(1)})$, $h\in H$, $b\in B$, where $\Delta(b)=b_{(1)}\otimes b_{(2)}$ in Sweedler's notation.

Fix a categorical completion $H\widehat{\otimes}H$ such that $(f\otimes 1)(H\widehat{\otimes}H)\subset H$ for all $f\in H^*_\mathscr{C}$. Equivalently, $\Phi_P:H^*_\mathscr{C}\to H$, $f\mapsto(f\otimes 1)(P)$ is a well-defined linear map. Denote $\mathscr{A}(H)$ the set of all $P\in H\widehat{\otimes}H$ such that $P\cdot(S^2\otimes 1)(\Delta(h))=\Delta(h)\cdot P$ for all $h\in H$. Clearly, $\mathscr{A}(H)$ is a subalgebra of $H\widehat{\otimes}H$. Elements of $\mathscr{A}(H)$ are analogous to M-matrices (see e.g. [13]). For $V\in C^{fin}$, set $c_V=c_{V,P}:=\Phi_P(z_V)\in\Phi_P((H^*_\mathscr{C})^H)$. Let $Z(H)$ be the center of H.

Theorem 1.5. *Let $P\in\mathscr{A}(H)$. Then $\Phi_P:H^*_\mathscr{C}\to H$ is a homomorphism of left H-modules, where H acts on $H^*_\mathscr{C}$ and H via $\diamond$ and ad, respectively. Moreover, $\Phi_P((H^*_\mathscr{C})^H)\subset Z(H)$ and the assignment $|V|\mapsto c_V$, $[V]\in\operatorname{Iso}\mathscr{C}^{fin}$ defines a homomorphism of abelian groups $\mathrm{ch}_\mathscr{C}:K_0(\mathscr{C}^{fin})\to Z(H)$.*

Surprisingly, Φ_P is often close to be an algebra homomorphism. To make this more precise, we generalize the notion of an algebra homomorphism as follows. Let A, B be $\Bbbk$-algebras and let $\mathscr{F}$ be a collection of subspaces in A. We say that a $\Bbbk$-linear map $\Phi:A\to B$ is a *$\mathscr{F}$-homomorphism* if $\Phi(U)\cdot\Phi(U')\subset\Phi(U\cdot U')$ for all $U,U'\in\mathscr{F}$. We say that $\mathscr{F}$ is multiplicative if $U\cdot U'\in\mathscr{F}$ for all $U,U'\in\mathscr{F}$. It is easy to see that $|\mathscr{F}|:=\sum_{U\in\mathscr{F}}U$ is a subalgebra of A and $\Phi(|\mathscr{F}|)$ is a subalgebra of B for any multiplicative family $\mathscr{F}$.

In what follows we denote $\mathscr{F}_\mathscr{C}$ the collection of all subspaces of H^* of the form H^*_V where $V\in\mathscr{C}$. By Theorem 1.4, $\mathscr{F}_\mathscr{C}$ is multiplicative.

Example 1.6. Let $H=\Bbbk G$ where G is a finite group and $\mathscr{C}$ be the category of its finite dimensional representations. Then the assignment $\delta_g\mapsto g^{-1}$ where $\delta_g(h)=\delta_{g,h}$, $g,h\in G$ defines an isomorphism of H-bimodules $\Phi:H^*\to H$. Let $\mathscr{F}_G=\{H^*_V\,:\,[V]\in\operatorname{Iso}^\circ\mathscr{C},\,\operatorname{Hom}_G(V,V\otimes V)\neq 0\}\subset\mathscr{F}_\mathscr{C}$. If $|G|\in\Bbbk^\times$ then Φ is an $\mathscr{F}_G$-homomorphism since $\Phi(H^*_V)\cdot\Phi(H^*_{V'})=0$ if $[V]\neq[V']\in\operatorname{Iso}^\circ\mathscr{C}$ and $\Phi(H^*_V)\cdot\Phi(H^*_V)=\Phi(H^*_V)$.

Denote by $\mathscr{M}(H)$ the set of all $P\in H\widehat{\otimes}H$ such that Φ_P is an $\mathscr{F}_\mathscr{C}$-homomorphism and by $\mathscr{M}_0(H)$ the set of all $P\in\mathscr{M}(H)$ such that Φ_P restricts to a homomorphism of algebras $(H^*_\mathscr{C})^H\to Z(H)$. We

abbreviate $H_{V,P} := \Phi_P(H_V^*)$ and $H_{\mathscr{C},P} := \Phi_P(H_{\mathscr{C}}^*) = \sum_{[V]\in \mathrm{Iso}\,\mathscr{C}} H_{V,P}$. Since $\mathscr{F}_{\mathscr{C}}$ is multiplicative, $H_{\mathscr{C},P}$ is a subalgebra of H for $P \in \mathscr{M}(H)$. The following is immediate.

Proposition 1.7. *Suppose that $P \in \mathscr{A}(H) \cap \mathscr{M}(H)$ and Φ_P is injective. Then:*

(a) *If $V \otimes_{D(V)} V^\star$ is a simple H-bimodule then it is isomorphic to $H_{V,P}$ as a left H-module;*

(b) *$H_{\mathscr{C},P} = \bigoplus_{[V]\in \mathrm{Iso}^\circ \mathscr{C}} H_{V,P}$ if $\mathscr{C}$ is semisimple and $V \otimes_{D(V)} V^\star$ is simple as an H-bimodule for each $V \in \mathscr{C}$ simple;*

(c) *If $P \in \mathscr{M}_0(H)$ then $\mathrm{ch}_{\mathscr{C}} : K_0(\mathscr{C}^{fin}) \to Z(H)$ is injective.*

The following theorem provides a sufficiently large subclass of $\mathscr{A}(H)\cap\mathscr{M}(H)$ and $\mathscr{A}(H)\cap\mathscr{M}_0(H)$.

Theorem 1.8. *Suppose that $P \in \mathscr{A}(H)$ such that $(\Delta \otimes 1)(P) = (m \otimes m \otimes 1)((T \otimes 1)P_{15}P_{35})$ for some $T \in H\widehat{\otimes}H\widehat{\otimes}H\widehat{\otimes}H$. Then $P \in \mathscr{M}(H)$. Moreover, if $(m^{op}\otimes m^{op})(T) = 1\otimes 1$ then $P \in \mathscr{M}_0(H)$.*

It should be noted that $\mathscr{M}(H)$ and $\mathscr{M}_0(H)$ are not exhausted by the above condition.

Example 1.9. Suppose that $\mathrm{char}\,\Bbbk \neq 2, 3$ and let $P_{\lambda,\mu} = \frac{1}{6}\sum_{\sigma\in S_3} 1\otimes\sigma + \frac{1}{36}\big[s_1 \otimes (1 + (2\mu-1)s_1 - (\mu+1)(s_2 + s_1s_2s_1) + s_1s_2 + s_2s_1)\big]_{S_3} + \frac{1}{18}\big[s_1s_2 \otimes (2 + (\lambda-1)s_1s_2 - (\lambda+1)s_2s_1)\big]_{S_3}$, where $\lambda, \mu \in \Bbbk$, $s_i = (i, i+1)$ and we abbreviate $\big[x\big]_G := \sum_{g\in G}(g \otimes g)x(g^{-1} \otimes g^{-1})$ for $x \in \Bbbk G \otimes \Bbbk G$. Then one can show that $P_{\lambda,\mu} \in \mathscr{A}(H) \cap \mathscr{M}_0(H)$ and that Φ_P is an isomorphism if and only if $(\lambda, \mu) \in (\Bbbk^\times)^2$. However, there is no $T \in H^{\otimes 4}$ such that the condition of Theorem 1.8 holds.

It turns out that $P \in \mathscr{A}(\Bbbk G)\cap\mathscr{M}_0(\Bbbk G)$ with Φ_P injective does not always exist for a given finite group G (for instance, it does not exist for dihedral groups different from $S_2 \times S_2$ and S_3) and thus it would be interesting to classify all finite groups G which admit such a P. Its existence provides a decomposition of $\Bbbk G$ into a direct sum of adjoint G-modules $H_{V,P}$ over all simple $\Bbbk G$-modules V (a mock Peter-Weyl decomposition) which is an alternative to the well-known Maschke decomposition into the direct sum of matrix algebras. As a further example, we constructed an 8-parameter family of such P for $G = S_4$. The answer is rather cumbersome (it involves 34 terms of the form $[g\otimes x]_{S_4}$, $g \in S_4$, $x \in \Bbbk S_4$) and is available at https://ishare.ucr.edu/jacobg/jdec-example.pdf).

Specializing Proposition 1.7 and Theorem 1.8 to quantized universal enveloping algebras we can recover Joseph's decomposition ([5]). Namely, let $H = U_q(\mathfrak{g})$ for a Kac-Moody algebra $\mathfrak{g}$ and $\mathscr{C}_{\mathfrak{g}}$ be the (semisimple) category of highest weight integrable $U_q(\mathfrak{g})$-modules (of type **1**, see e.g. [1]); then $V^\star$ is the graded dual. Let Λ^+ be the monoid of dominant weights for $\mathfrak{g}$ and denote $V(\lambda)$ a highest weight simple integrable module of highest weight $\lambda \in \Lambda^+$. We construct $P = P_{\mathfrak{g}}$ with $\Phi_{P_{\mathfrak{g}}}$ injective in Lemma 2.9 and obtain the following Theorem which refines results of [5].

Theorem 1.10. (a) *For $\lambda \in \Lambda^+$, $H_{V(\lambda),P} = \mathrm{ad}\,U_q(\mathfrak{g})(K_{2\lambda}) \cong V(\lambda) \otimes V(\lambda)^\star$.*

(b) *$\sum_{\lambda\in\Lambda^+} \mathrm{ad}\,U_q(\mathfrak{g})(K_{2\lambda})$ is direct and is a subalgebra of $U_q(\mathfrak{g})$.*

Furthermore, part (c) of Proposition 1.7, which generalizes a classic result of Drinfeld ([2]), yields

Theorem 1.11. *Let $\mathfrak{g}$ be semisimple. Then the assignment $|V| \mapsto c_V$ defines an isomorphism of algebras $\mathbb{Q}(q) \otimes_{\mathbb{Z}} K_0(\mathfrak{g}-\mathrm{mod}) \to Z(U_q(\mathfrak{g}))$.*

This provides the following refinements of classic results of Duflo, Harish-Chandra and Rosso ([10]).

Corollary 1.12. *For $\mathfrak{g}$ semisimple, $Z(U_q(\mathfrak{g}))$ is freely generated by the $c_{V(\omega)}$ where the ω are fundamental weights of $\mathfrak{g}$, and $c_{V(\lambda)}c_{V(\mu)} = \sum_{\nu\in\Lambda^+}[V(\lambda) \otimes V(\mu) : V(\nu)]c_{V(\nu)}$ for any $\lambda, \mu \in \Lambda^+$.*

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2. Notation and proofs

Recall that, given an H-bimodule B, B^* is naturally an H-bimodule via $(h \triangleright f \triangleleft h')(b) = f(h' \triangleright b \triangleleft h)$, $f \in B^*$, $h, h' \in H$, $b \in B$. In particular, H^* is an H-bimodule.

2.1. **Proof of Theorem 1.3.** The following are immediate.

Lemma 2.1. $\langle V, W^\star\rangle_{V\oplus W} = 0 = \langle W, V^\star\rangle_{V\oplus W}$.

Lemma 2.2. *Let V, W be left H-modules and let $\rho : H \otimes_{\Bbbk} W \to V$ be a $\Bbbk$-linear map. Then:*

(a) *the assignment $h \triangleright_\rho (v, w) = (h \triangleright v + \rho(h \otimes w), h \triangleright w)$, $h \in H$, $v \in V$, $w \in W$, defines a left H-module structure $V \oplus_\rho W$ on $V \oplus W$ if and only if*

$$\rho(hh' \otimes w) = \rho(h \otimes h' \triangleright w) + h \triangleright \rho(h' \otimes w), \qquad h, h' \in H,\ w \in W. \tag{2.1}$$

In that case V is an H-submodule of $V \oplus_\rho W$ and $W = (V \oplus_\rho W)/V$.

(b) *A short exact sequence of H-modules $0 \to V \to U \to W \to 0$ is equivalent to $0 \to V \to V \oplus_\rho W \to W \to 0$ for some ρ satisfying (2.1).*

Thus, given $V \subset U$ in $\mathscr{C}$, we can replace the natural short exact sequence $0 \to V \to U \to U/V \to 0$ by the one from Lemma 2.2.

Lemma 2.3. *Let V, W be left H-modules and let ρ be as in Lemma 2.2. Then $\beta_{V\oplus_\rho W}(x + y) = \beta_V(x) + \beta_V(y)$ for any $x \in V \otimes V^\star$, $y \in W \otimes W^\star$.*

Proof. It suffices to verify the assertion for $x = v \otimes f$ and $y = w \otimes g$, $v \in V$, $w \in W$, $f \in V^\star$, $g \in W^\star$. We have by Lemmata 2.1, 2.2(a)

$$\begin{aligned}\beta_{V\oplus_\rho W}(v \otimes f + w \otimes g)(h) &= \langle h \triangleright_\rho v \otimes f + h \triangleright_\rho w \otimes g\rangle_{V\oplus W}\\ &= \langle h \triangleright v, f\rangle_V + \langle \rho(h \otimes w), f\rangle_{V\oplus W} + \langle h \triangleright w, g\rangle_W = \beta_V(v \otimes f)(h) + \beta_W(w \otimes g)(h).\end{aligned}$$ □

Since $1_{V\oplus_\rho W} = 1_V + 1_W$ where $1_V \in V \otimes V^\star$, $1_W \in W \otimes W^\star$, it follows from Lemma 2.3 that $z_{V\oplus_\rho W} = z_V + z_W$ and the map $K_0(\mathscr{C}) \to \mathcal{Z}_{\mathscr{C}}$, $|V| \mapsto z_V$ is a well-defined surjective homomorphism of abelian groups. Also, $z_V \in \sum_{[S]\in \mathrm{Iso}^\circ \mathscr{C}} \mathbb{Z} z_S$ for each $V \in \mathscr{C} = \mathscr{C}^{fin}$ because it has finite length. Since the set $\{z_V\}_{[V]\in\mathrm{Iso}^\circ \mathscr{C}} \subset \underline{H}^*_{\mathscr{C}}$ is $\Bbbk$-linearly independent by Proposition 1.1(d), the injectivity follows. □

2.2. **Algebra structure on $H^*_{\mathscr{C}}$.** Henceforth we assume that $H = (H, m, \Delta, \varepsilon)$ is a bialgebra.

Then H^* is a unital algebra with the multiplication defined by $(\phi \cdot \xi)(h) = \phi(h_{(1)})\xi(h_{(2)})$, $h \in H$, $\phi, \xi \in H^*$, $\Delta(h) = h_{(1)} \otimes h_{(2)}$ in Sweedler notation and the unity is ε.

Lemma 2.4. *$(H^*)^H$ is a subalgebra of H^*.*

Proof. Observe that $\phi \in (H^*)^H$ if and only if $\phi(hh') = \phi(h'h)$ for all $h, h' \in H$. Then, given $h, h' \in H$ and $\xi, \xi' \in (H^*)^H$ we have

$$(\xi \cdot \xi')(hh') = \xi(h_{(1)}h'_{(1)})\xi'(h_{(2)}h'_{(2)}) = \xi(h'_{(1)}h_{(1)})\xi'(h'_{(2)}h_{(2)}) = (\xi \cdot \xi')(h'h).$$ □

Proof of Theorem 1.4. Note that in the category of $\Bbbk$-vector spaces there is a natural isomorphism $\kappa : (V \otimes V^\star) \otimes (V' \otimes V'^\star) \to (V \otimes V') \otimes (V \otimes V')^\star$, $\kappa(v \otimes f \otimes v' \otimes f') = v \otimes v' \otimes f' \otimes f$, $v \in V$, $v' \in V'$, $f \in V^\star$, $f' \in V'^\star$. Then, clearly, $\langle\cdot,\cdot\rangle_{V\otimes V'} \circ \kappa = \langle\cdot,\cdot\rangle_V \otimes \langle\cdot,\cdot\rangle_{V'}$ which immediately implies that $\tilde\beta_V \otimes \tilde\beta_{V'} = \tilde\beta_{V\otimes V'} \circ \kappa$ where $\tilde\beta_U := \beta_U \circ \pi_U$ where $\pi_U : U \otimes_{\Bbbk} U^\star \to U \otimes_{D(U)} U^\star$ is the natural projection. This proves the first assertion and also the second once we observe that $1_{V\otimes V'} = \kappa(1_V \otimes 1_{V'})$. □

2.3. **The Hopf algebra case.** Suppose now that $H = (H, m, \Delta, \varepsilon, S)$ is a Hopf algebra. Since H is naturally an H-bimodule, $\operatorname{ad} : H \to \operatorname{End}_{\Bbbk} H$ is a homomorphism of algebras. We also define $\operatorname{ad}^* : H^{op} \to \operatorname{End}_{\Bbbk} H$ by $(\operatorname{ad}^* h)(h') = S(h_{(1)})h'S^2(h_{(2)})$, which is a homomorphism of algebras. Henceforth, given $a \in H^{\otimes n}$ we write it in Sweedler-like notation as $a = a_1 \otimes \cdots \otimes a_n$ with summation understood.

Proof of Theorem 1.5. We need the following equivalent descriptions of $\mathscr{A}(H)$.

Lemma 2.5. *Let $P = P_1 \otimes P_2 \in H \widehat{\otimes} H$. The following are equivalent:*

(a) $P \cdot (S^2 \otimes 1) \circ \Delta(h) = \Delta(h) \cdot P$;

(b) $(1 \otimes h) \cdot P = (\operatorname{ad}^* h_{(1)})(P_1) \otimes P_2 h_{(2)}$;

(c) $(\operatorname{ad}^* h \otimes 1)(P) = (1 \otimes \operatorname{ad} h)(P)$.

Proof. By (a) we have $h_{(1)} \otimes P_1 S^2(h_{(2)}) \otimes P_2 h_{(3)} \otimes h_{(4)} = h_{(1)} \otimes h_{(2)} P_1 \otimes h_{(3)} P_2 \otimes h_{(4)}$ for all $h \in H$. Then (b) and (c) follow by applying $m(S \otimes 1) \otimes 1 \otimes \varepsilon$ and $m(S \otimes 1) \otimes m(1 \otimes S)$, respectively, to both sides. Part (b) implies (a) since $h_{(1)}(\operatorname{ad}^* h_{(2)})(h') = h'S^2(h)$. Finally, (c) implies (b) since $(\operatorname{ad}^* h_{(1)})(P_1) \otimes P_2 h_{(2)} = P_1 \otimes \operatorname{ad} h_{(1)}(P_2) h_{(2)} = P_1 \otimes h P_2$. $\square$

Lemma 2.6. *Let B be an H-bimodule and set $B^{\diamond H} := \{b \in B : h \diamond b = \varepsilon(h)b,\ h \in H\}$. Then $B^H \subset B^{\diamond H} \subset B^{S(H)}$ with the equality if S is invertible.*

Proof. Let $h \in H$. Then for all $b \in B^H$ we have $h \diamond b = S^2(h_{(2)}) \triangleright b \triangleleft S(h_{(1)}) = S^2(h_{(2)})S(h_{(1)}) \triangleright b = S(h_{(1)}S(h_{(2)})) \triangleright b = \varepsilon(h)b$. On the other hand, for all $b \in H^{\diamond H}$, $S(h) \triangleright b = \varepsilon(h_{(1)})S(h_{(2)}) \triangleright m = S(h_{(3)})S^2(h_{(2)}) \triangleright m \triangleleft S(h_{(1)}) = S(S(h_{(2)})h_{(3)}) \triangleright m \triangleleft S(h_{(1)}) = m \triangleleft S(h)$. $\square$

The following Lemma is well-known and can be proved similarly.

Lemma 2.7. $Z(H) = H^H = H^{\operatorname{ad} H} := \{h' \in H : (\operatorname{ad} h)(h') = \varepsilon(h)h',\ h \in H\}$. $\square$

By Lemma 2.5(c) we have, for all $h \in H$, $\xi \in H^*_{\mathscr{C}}$

$$\Phi_P(h \diamond \xi) = (S^2(h_{(2)}) \triangleright \xi \triangleleft S(h_{(1)}))(P_1)P_2 = \xi((\operatorname{ad}^* h)P_1)P_2 = \xi(P_1)(\operatorname{ad} h)(P_2) = (\operatorname{ad} h)\Phi_P(\xi).$$

Furthermore, if $\xi \in (H^*_{\mathscr{C}})^H$ then $\Phi_P(h \diamond \xi) = \varepsilon(h)\Phi_P(\xi) = (\operatorname{ad} h)\Phi_P(\xi)$, whence $\Phi_P(\xi) \in Z(H)$. $\square$

Proof of Theorem 1.8. Suppose that P satisfies $(\Delta \otimes 1)(P) = t_1 P_1 t_2 \otimes t_3 P_1' t_4 \otimes P_2 P_2'$, for some $T = t_1 \otimes t_2 \otimes t_3 \otimes t_4 \in H^{\widehat{\otimes} 4}$ where $P = P_1 \otimes P_2 = P_1' \otimes P_2'$. Then for any $\xi, \xi' \in H^*_{\mathscr{C}}$

$$\begin{aligned}\Phi_P(\xi \cdot \xi') = (\xi \cdot \xi')(P_1)P_2 = \xi(t_1 P_1 t_2)\xi'(t_3 P_1' t_4)P_2 P_2' &= (t_2 \triangleright \xi \triangleleft t_1)(P_1)(t_4 \triangleright \xi' \triangleleft t_3)(P_1')P_2 P_2' \\ &= \Phi_P(t_2 \triangleright \xi \triangleleft t_1) \cdot \Phi_P(t_4 \triangleright \xi' \triangleleft t_3).\end{aligned} \tag{2.2}$$

Take $\xi \in H^*_V$, $\xi' \in H^*_{V'}$. Then $\xi \cdot \xi' \in H^*_{V \otimes V'}$ by Theorem 1.4(a) and $\Phi_P(\xi \cdot \xi') \in H_{V,P} \cdot H_{V',P}$ by (2.2). Therefore, $P \in \mathscr{M}(H)$. Furthermore, assume that $t_2 t_1 \otimes t_4 t_3 = 1 \otimes 1$, and let $\xi, \xi' \in (H^*_{\mathscr{C}})^H$. Then (2.2) yields $\Phi_P(\xi \cdot \xi') = \Phi_P(t_2 t_1 \triangleright \xi) \cdot \Phi_P(t_4 t_3 \triangleright \xi') = \Phi_P(\xi) \cdot \Phi_P(\xi')$. This implies that $P \in \mathscr{M}_0(H)$. $\square$

2.4. **Applications.** Let $\mathscr{R}(H)$ be the set of pairs (R^+, R^-), $R^\pm \in H \widehat{\otimes} H$, such that $R^+_{21} R^- \cdot \Delta(h) = \Delta(h) \cdot R^+_{21} R^-$ for all $h \in H$ and $(\Delta \otimes 1)(R^\pm) = R^\pm_{13} R^\pm_{23}$, $(1 \otimes \Delta)(R^+) = R^+_{13} R^+_{12}$. Clearly, $(R, R) \in \mathscr{R}(H)$ if R is an R-matrix for H.

Lemma 2.8. *Suppose that there exists $\mathbf{g} \in H$ group-like such that $\mathbf{g}S^2(h) = h\mathbf{g}$ for all $h \in H$. Let $(R^+, R^-) \in \mathscr{R}(H)$. Then $P := R^+_{21} \cdot R^- \cdot (\mathbf{g} \otimes 1) \in \mathscr{A}(H) \cap \mathscr{M}_0(H)$.*

Proof. Write $R^\pm = r_1^\pm \otimes r_2^\pm = s_1^\pm \otimes s_2^\pm$. Since $R_{21}^+ R^- \cdot \Delta(h) = \Delta(h) \cdot R_{21}^+ R^-$ we have

$$P \cdot (S^2 \otimes 1)(\Delta(h)) = r_2^+ r_1^- \mathbf{g} S^2(h_{(1)}) \otimes r_1^+ r_2^- h_{(2)} = r_2^+ r_1^- h_{(1)} \mathbf{g} \otimes r_1^+ r_2^- h_{(2)} = \Delta(h) \cdot P.$$

Thus, $P \in \mathscr{A}(H)$. Furthermore, $(\Delta \otimes 1)(P) = R_{32}^+ R_{31}^+ R_{13}^- R_{23}^- (\mathbf{g} \otimes \mathbf{g} \otimes 1) = P_1 \otimes r_2^+ r_1^- \mathbf{g} \otimes r_1^+ P_2 r_2^-$. Since $(\Delta \otimes 1)(R^+) = r_1^+ \otimes s_1^+ \otimes r_1^+ s_1^+$, by Lemma 2.5(b) we obtain

$$\begin{aligned}(\Delta \otimes 1)(P) = (\operatorname{ad}^* r_1^+)(P_1) \otimes r_2^+ s_2^+ r_1^- \mathbf{g} \otimes P_2 s_1^+ r_2^- &= (\operatorname{ad}^* r_1^+)(P_1) \otimes r_2^+ P_1' \otimes P_2 P_2' \\ &= S(r_1^+) P_1 S^2(s_1^+) \otimes r_2^+ s_2^+ P_1' \otimes P_2 P_2'.\end{aligned}$$

Thus, $P \in \mathscr{M}(H)$ with $T = (S \otimes S^2 \otimes 1 \otimes 1)(R_{13}^+ \cdot R_{23}^+)$. Finally, $(m^{op} \otimes m^{op})(T) = S^2(s_2^+) S(r_1^+) \otimes r_2^+ s_2^+ = (S \otimes 1)(R^+ \cdot (S \otimes 1)(R^+)) = 1 \otimes 1$. Thus, $P \in \mathscr{M}_0(H)$. □

If P is as in Lemma 2.8 we obtain

$$\Phi_P(\beta_V(v \otimes f)) = r_1^+ \langle r_2^+ r_1^- \mathbf{g} \triangleright v, f\rangle_V r_2^- = r_1^+ \langle r_1^- \triangleright \mathbf{g}(v), f \triangleleft r_2^+\rangle_V r_2^-, \qquad v \in V,\ f \in V^\star. \tag{2.3}$$

Let $\Bbbk = \mathbb{Q}(q)$ and let $U_q(\mathfrak{g})$ be a quantized enveloping algebra corresponding to a symmetrizable Kac-Moody algebra $\mathfrak{g}$ which is a Hopf algebra generated by E_i, F_i, $i \in I$ and K_μ, $\mu \in \Lambda$, where Λ is a weight lattice of $\mathfrak{g}$, with $\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_{\alpha_i}$, $\Delta(F_i) = F_i \otimes 1 + K_{-\alpha_i} \otimes F_i$, $\Delta(K_\mu) = K_\mu \otimes K_\mu$, $\varepsilon(E_i) = \varepsilon(F_i) = 0$ and $\varepsilon(K_\mu) = 1$, where α_i, $i \in I$ are simple roots of $\mathfrak{g}$. Let $\mathcal{K}$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the K_μ, $\mu \in \Lambda$. After [2, 8], there exists a unique R-matrix in a weight completion $U_q(\mathfrak{g}) \widehat{\otimes} U_q(\mathfrak{g})$ of the form $R = R_0 R_1$ where $R_1 \in U_q^+(\mathfrak{g}) \widehat{\otimes} U_q^-(\mathfrak{g})$ is essentially Θ^{op} in the notation of [8] and satisfies $(\varepsilon \otimes 1)(R_1) = (1 \otimes \varepsilon)(R_1) = 1 \otimes 1$, while $R_0 \in \mathcal{K} \widehat{\otimes} \mathcal{K}$ is determined by the following condition: for any $\mathcal{K}$-modules $V^\pm$ such that $K_\mu|_{V^\pm} = q^{(\mu,\mu_\pm)} \operatorname{id}_{V^\pm}$, $\mu, \mu_\pm \in \Lambda$, we have $R_0|_{V^- \otimes V^+} = q^{(\mu_-,\mu_+)} \operatorname{id}_{V^- \otimes V^+}$. Here $(\cdot,\cdot)$ is the Kac-Killing form on $\Lambda \times \Lambda$ ([6]). The following is immediate.

Lemma 2.9. *Let $R = r_1 \otimes r_2$ be as above. Let $v_\lambda \in V(\lambda)$ ($f_\lambda \in V(\lambda)^\star$) be a highest (respectively, lowest) weight vector of weight λ (respectively, $-\lambda$), $\lambda \in \Lambda^+$. Then $r_1 \triangleright v_\lambda \otimes r_2 = v_\lambda \otimes K_\lambda$ and $r_1 \otimes f_\lambda \triangleleft r_2 = K_\lambda \otimes f_\lambda$.* □

Proof of Theorem 1.10. Since $V(\lambda)$ is a simple highest weight module, $D(V(\lambda)) \cong \Bbbk$. Note that for any $\lambda, \mu \in \Lambda^+$, $V(\lambda) \otimes V(\mu)$ is a simple $U_q(\mathfrak{g} \oplus \mathfrak{g}) = U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$-module of highest weight (λ, μ). Twisting $V(\mu)$ with the anti-automorphism of $U_q(\mathfrak{g})$ interchanging F_i and E_i, we conclude that $V(\lambda) \otimes V(\lambda)^\star$ is a simple $U_q(\mathfrak{g})$-bimodule. Taking into account that $\mathbf{g} = K_{-2\rho}$ we obtain from Lemma 2.9 and (2.3) that $\Phi_P(\beta_{V(\lambda)}(v_\lambda \otimes f_\lambda)) = K_\lambda \langle \mathbf{g} \triangleright v_\lambda, f_\lambda\rangle K_\lambda \in \Bbbk^\times K_{2\lambda}$. Since $V(\lambda) \otimes V(\lambda)^\star$ is cyclic on $v_\lambda \otimes f_\lambda$ as $U_q(\mathfrak{g})$-module with the $\diamond$ action, $H_{V(\lambda)}$ is cyclic on $K_{2\lambda}$ as the $\operatorname{ad} U_q(\mathfrak{g})$-module by the above. Since $\beta_{V(\lambda)}$ is injective by Theorem 1.1(c) and Φ_P is injective by [2], it follows that $H_{V(\lambda)} \cong V(\lambda) \otimes V(\lambda)^\star$. This proves (a). Then the sum in (b) is direct by Proposition 1.7(b) and coincides with $H_{\mathscr{C}_\mathfrak{g},P}$ which is always a subalgebra of H. □

Proof of Theorem 1.11. Since $D(V(\lambda)) \cong \Bbbk$, Theorem 1.10 implies that $Z(H_{\mathscr{C}_\mathfrak{g},P_\mathfrak{g}}) = \bigoplus_{\lambda \in \Lambda^+} \Bbbk c_{V(\lambda)}$, hence the assignment $|V(\lambda)| \mapsto c_{V(\lambda)}$ is an isomorphism $\Bbbk \otimes_\mathbb{Z} K_0(\mathscr{C}_\mathfrak{g}) \to \Phi_{P_\mathfrak{g}}((H^*_{\mathscr{C}_\mathfrak{g}})^H) = Z(H_{\mathscr{C}_\mathfrak{g},P_\mathfrak{g}})$ as in Proposition 1.7(c). By [7], $K_0(\mathscr{C}_\mathfrak{g}) = K_0(\mathfrak{g} - \mathrm{mod})$ where $\mathfrak{g} - \mathrm{mod}$ is the category of finite dimensional $\mathfrak{g}$-modules. On the other hand, each non-zero element of $Z(U_q(\mathfrak{g}))$ is ad-invariant, hence generates a one-dimensional $\operatorname{ad} U_q(\mathfrak{g})$-module and thus is contained in $H_{\mathscr{C}_\mathfrak{g},P_\mathfrak{g}}$ by [5]. Therefore, $Z(U_q(\mathfrak{g})) \subset H_{\mathscr{C}_\mathfrak{g},P_\mathfrak{g}}$ hence $Z(U_q(\mathfrak{g})) = Z(H_{\mathscr{C}_\mathfrak{g},P_\mathfrak{g}})$. □

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